Correlation Kernels Determine Local Symmetry and Projective Dynamics

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A multipartite factorization separates traceless observables into one-site and correlation sectors. We solve the inverse problem of recognizing the correlation sector without assuming the factorization. Let $\mathcal{X} \subseteq \mathrm{Herm}_0(\mathbb{C}^N)$ be an arbitrary real subspace, set $\mathfrak{m} = i\mathcal{X}$, and let $\mathfrak{k}$ be its Lie normalizer in $\mathfrak{su}(N)$. The subspace $\mathcal{X}$ is the joint kernel of all one-site partial traces for a factorization of type $N = \prod_r q_r$ exactly when $\mathfrak{m} = \mathfrak{k}^\perp$ and $\mathfrak{k} \cong \bigoplus_r \mathfrak{su}(q_r)$. The simple ideals of $\mathfrak{k}$ then recover the local matrix algebras. The same kernel is the kernel of the linear marginal map and the direction space of every full-rank equal-marginal state fibre. It also gives the tangent space to factorization space and the horizontal sector of its unique invariant connection. That connection has full local projective holonomy. The skew-Hermitian kernel generates $\mathfrak{su}(N)$ at bracket step two. For every pure ray, fully product factorizations of the fixed type form a measure-zero submanifold. No physical control law or spacetime claim is derived.

1 Introduction

A tensor-product structure specifies local observable algebras, partial traces, reduced states, product operations, and entanglement predicates. A bare finite-dimensional Hilbert space selects none of these. Existing constructions start from accessible commuting algebras, an operator partition, a Hamiltonian or its spectrum, or a trial family of factorizations [1, 9, 10, 15, 20–22]. Embedded local Lie algebras and their commutants also have established roles in quantum control [24]. Pure-state separability is consequently a relation between a ray and a subsystem structure [6].

Recent work represents a tensor-product structure by its one-local operator subspace and measures unitary displacement by Hilbert–Schmidt projector distance [3]. Their zero-distance classification already gives the full stabilizer as local unitaries composed with permutations among equal-dimensional factors. Orthogonal complementation therefore gives the same full stabilizer for the correlation kernel. The projected interaction component also defines the infinitesimal tensor-product-structure metric [23]. Related horizontal distributions occur in geometric quantum control and in the KP problem on compact symmetric spaces [14, 19].

The homogeneous-space and holonomy ingredients are standard [2, 13, 17]. Lie generation is standard [4, 11, 12, 18]. The result not contained in the cited constructions is an intrinsic recognition criterion. It begins with an arbitrary global operator subspace, computes its Lie normalizer, and decides whether the subspace is exactly a multipartite one-site marginal kernel. Under the equivalent recognition conditions, the same reductive pair gives the tensor-product-structure tangent bundle, invariant connection, curvature, holonomy, and step-two projective Lie closure. Recent subsystem geometry formulated through Segre stabilizers and operator partitions does not give this normalizer-and-orthogonality criterion [7, 16].

Projective dynamics means Lie closure in $\mathfrak{su}(N)$. It does not mean that the recognized horizontal directions are physically available controls.

2 Intrinsic correlation data

Let $\mathcal{H} \cong \mathbb{C}^N$ with $N \geq 4$. Scalar unitary phase acts trivially by conjugation, so the global transformation group is

$$G = \mathrm{PU}(N) = \mathrm{U}(N)/\,\mathrm{U}(1).$$

Its Lie algebra is

$$\mathfrak{g} = \mathfrak{su}(N),$$

where $\mathfrak{su}(N)$ denotes the traceless skew-Hermitian operators. This convention matters because Hermitian operators are multiplied by i before they are treated as infinitesimal unitary generators. For $g \in G$, the adjoint action is $\mathrm{Ad}_g A = UAU^\dagger$, where U is any unitary representative of the projective class g. The result is independent of the representative.

$\mathrm{Herm}_0(\mathcal{H})$ denotes the real vector space of traceless Hermitian operators on $\mathcal{H}$. It carries the Hilbert–Schmidt inner product

$$\langle A, B\rangle_{\mathrm{HS}} = \mathrm{Tr}(AB).$$

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The corresponding positive inner product on $\mathfrak{g}$ is $\langle X, Y\rangle_{\mathfrak{g}} = -\operatorname{Tr}(XY)$. For a subspace of either ambient real inner-product space, the superscript $\perp$ denotes its orthogonal complement in that ambient space.

Let $\mathcal{X} \subseteq \mathrm{Herm}_0(\mathcal{H})$ be an arbitrary real subspace. This is the input to the inverse problem. It is not assumed to arise from subsystems, partial traces, or a tensor product. Write the same subspace in skew-Hermitian generator form as

$$\mathfrak{m} = i\mathcal{X}, \qquad \mathfrak{k} = \mathfrak{n}_{\mathfrak{g}}(\mathfrak{m}) = \{Z \in \mathfrak{g} : [Z, \mathfrak{m}] \subseteq \mathfrak{m}\}. \quad (1)$$

Multiplication by i converts Hermitian operators into infinitesimal unitary generators. The Lie normalizer $\mathfrak{k}$ contains exactly those global generators whose commutators preserve $\mathfrak{m}$. Explicitly, $[Z, \mathfrak{m}]$ is the set of commutators $[Z, M]$ with $M \in \mathfrak{m}$. The space $\mathfrak{m}$ is only a real vector subspace at this stage. It is not assumed to be a Lie algebra. Equation (1) extracts the continuous symmetry algebra of the supplied subspace without presupposing what that symmetry represents.

3 Recognition of correlation kernels

3.1 Kernel of a known factorization

Fix integers $m \geq 2$ and $q_r \geq 2$ with $N = \prod_{r=1}^{m} q_r$. A supplied factorization of this type is a unitary identification

$$\mathcal{H} = \bigotimes_{r=1}^{m} \mathcal{H}_r, \qquad \dim \mathcal{H}_r = q_r.$$

Write $[m] = \{1, \ldots, m\}$ and let $\bar{r} = [m] \setminus \{r\}$ denote the complementary set of sites. The operators $\mathbf{1}_r$ and $\mathbf{1}_{\bar{r}}$ are the identities on $\mathcal{H}_r$ and on $\bigotimes_{s\in\bar{r}} \mathcal{H}_s$. We write $\operatorname{Tr}_{\bar{r}}$ for the partial trace over the complementary factors, so its output is an operator on $\mathcal{H}_r$.

For $A \in \mathrm{End}(\mathcal{H}_r)$, define its embedding into the global operator algebra by

$$\iota_r(A) = \mathbf{1}_1 \otimes \cdots \otimes \mathbf{1}_{r-1} \otimes A \otimes \mathbf{1}_{r+1} \otimes \cdots \otimes \mathbf{1}_m.$$

We write $A^{(r)} = \iota_r(A)$. Operators with different site labels commute, and their ordinary product is the corresponding tensor-product operator.

We call the joint kernel of these one-site marginal maps the *correlation kernel* of the factorization:

$$\mathcal{C} = \{A \in \mathrm{Herm}_0(\mathcal{H}) : \operatorname{Tr}_{\bar{r}} A = 0 \text{ for every } r\}.$$

It contains exactly the traceless operator directions invisible to all one-site marginals. Its orthogonal complement is the one-site sector

$$\mathcal{L}_{\mathrm{loc}} = \bigoplus_{r=1}^{m} \iota_r\big(\mathrm{Herm}_0(\mathcal{H}_r)\big).$$

Consequently

$$\mathcal{C} = \bigcap_{r=1}^{m} \ker \operatorname{Tr}_{\bar{r}}, \qquad \mathrm{Herm}_0(\mathcal{H}) = \mathcal{L}_{\mathrm{loc}} \oplus \mathcal{C}. \quad (2)$$

For two qubits, the split has the concrete form

$$\mathcal{L}_{\mathrm{loc}} = \mathrm{span}_{\mathbb{R}}\{\sigma_j \otimes \mathbf{1},\ \mathbf{1} \otimes \sigma_k\},$$
$$\mathcal{C} = \mathrm{span}_{\mathbb{R}}\{\sigma_j \otimes \sigma_k\}, \qquad 1 \leq j, k \leq 3.$$

Here $\sigma_1, \sigma_2, \sigma_3$ are the Pauli matrices, and the indices j and k range independently over $\{1, 2, 3\}$. The six one-site Pauli operators and the nine two-site Pauli products exhaust the fifteen-dimensional real space $\mathrm{Herm}_0(\mathbb{C}^4)$. The second subspace is therefore not an added class of operators. It is the full orthogonal complement of the first.

The factorization was used only to define the target object. The inverse problem starts with the arbitrary subspace $\mathcal{X}$ from Section 2 and asks whether it is the correlation kernel of any factorization of the stated type.

3.2 Intrinsic criterion

For a unitary identification F and a subspace $\mathcal{S} \subseteq \mathrm{Herm}_0(\mathcal{H})$, the notation $F^\dagger \mathcal{S} F$ means the conjugated subspace $\{F^\dagger A F : A \in \mathcal{S}\}$.

Theorem 3.1. For the fixed factor type $\boldsymbol{q} = (q_1, \ldots, q_m)$, the following statements are equivalent.

(a) There is a unitary identification $F : \bigotimes_{r=1}^{m} \mathbb{C}^{q_r} \to \mathcal{H}$ for which $F^\dagger \mathcal{X} F$ is the joint kernel of the one-site partial traces on $\mathrm{Herm}_0(\bigotimes_r \mathbb{C}^{q_r})$.

(b) The intrinsic spaces in Eq. (1) satisfy

$$\mathfrak{m} = \mathfrak{k}^{\perp}, \qquad \mathfrak{k} \cong \bigoplus_{r=1}^{m} \mathfrak{su}(q_r) \quad (3)$$

as compact real Lie algebras.

When these conditions hold, the simple ideals of $\mathfrak{k}$ generate the local full matrix algebras. The factorization is therefore recovered from $\mathcal{X}$ up to local unitary frames and permutations of equal-dimensional factors.

Condition (a) is a concrete tensor-product statement. Condition (b) uses only the supplied subspace, the Hilbert–Schmidt metric, and its Lie normalizer. In Eq. (3), the orthogonal complement is taken inside $\mathfrak{g}$. The equality $\mathfrak{m} = \mathfrak{k}^{\perp}$ says that the supplied subspace exhausts every direction orthogonal to its own normalizer. The Lie-algebra isomorphism says that this normalizer has exactly the continuous algebra of an m-part local unitary group.

Proof. Forward implication. Assume statement (a) and use F to identify the tensor factors. Under this identification, $\mathcal{X} = \mathcal{C}$. Set

$$V_S = \operatorname{span}_{\mathbb{R}} \left\{ \prod_{r \in S} A_r^{(r)} : A_r \in \operatorname{Herm}_0(\mathcal{H}_r) \right\}$$

for every nonempty set of sites $S \subseteq [m]$. A tensor product of local Hermitian bases gives the following orthogonal decomposition by operator support:

$$\operatorname{Herm}_0(\mathcal{H}) = \bigoplus_{\varnothing \neq S \subseteq [m]} V_S,$$
$$\mathcal{L}_{\mathrm{loc}} = \bigoplus_{|S|=1} V_S, \qquad \mathcal{C} = \bigoplus_{|S| \geq 2} V_S.$$

The set S records the factors on which an operator is traceless. A term survives a one-site partial trace exactly when its support is that one site. This proves Eq. (2) directly.

Set $\mathfrak{k}_{\mathrm{loc}} = i\mathcal{L}_{\mathrm{loc}}$. Equations (1) and (2) give $\mathfrak{m} = \mathfrak{k}_{\mathrm{loc}}^{\perp}$. Local conjugations preserve operator support, so $\mathfrak{k}_{\mathrm{loc}} \subseteq \mathfrak{n}_{\mathfrak{g}}(\mathfrak{m})$.

Let Z belong to this normalizer and define $\operatorname{ad}_Z(Y) = [Z, Y]$. This linear map is skew-adjoint for the Hilbert–Schmidt inner product. Since it preserves $\mathfrak{m}$, it also preserves $\mathfrak{m}^{\perp} = \mathfrak{k}_{\mathrm{loc}}$. Its restriction to the semisimple algebra $\mathfrak{k}_{\mathrm{loc}}$ is a derivation and is therefore inner. There is $A \in \mathfrak{k}_{\mathrm{loc}}$ such that

$$[Z, Y] = [A, Y] \quad \text{for every } Y \in \mathfrak{k}_{\mathrm{loc}}.$$

The difference $Z - A$ centralizes every local matrix algebra. These algebras generate $\operatorname{End}(\mathcal{H})$, so their commutant contains only scalars. No nonzero scalar lies in $\mathfrak{su}(N)$. Hence $Z = A$, and

$$\mathfrak{n}_{\mathfrak{g}}(\mathfrak{m}) = \mathfrak{k}_{\mathrm{loc}} \cong \bigoplus_{r=1}^{m} \mathfrak{su}(q_r).$$

This proves statement (b).

Reverse implication. Assume statement (b), without assuming a factorization. If an element of $\mathfrak{g}$ centralizes $\mathfrak{k}$, it preserves $\mathfrak{k}^{\perp} = \mathfrak{m}$ and therefore belongs to $\mathfrak{n}_{\mathfrak{g}}(\mathfrak{m}) = \mathfrak{k}$. The center of the semisimple algebra $\mathfrak{k}$ is zero, so

$$\mathfrak{z}_{\mathfrak{g}}(\mathfrak{k}) := \{Z \in \mathfrak{g} : [Z, Y] = 0 \text{ for every } Y \in \mathfrak{k}\} = \{0\}.$$

The representation of $\mathfrak{k}$ on $\mathcal{H}$ is consequently irreducible. Otherwise an invariant orthogonal projection P would make $i(P - (\operatorname{Tr} P/N)\mathbf{1})$ a nonzero element of the centralizer.

Write $\mathfrak{k} = \bigoplus_r \mathfrak{k}_r$ according to its simple ideals. An irreducible complex module for a direct sum of compact simple Lie algebras is an outer tensor product

$$\mathcal{H} \cong \bigotimes_{r=1}^{m} W_r, \qquad d_r = \dim W_r, \qquad N = \prod_{r=1}^{m} d_r.$$

Here W_r carries the irreducible action of the rth simple ideal, the other ideals act trivially on W_r, and the full algebra acts factorwise on the displayed tensor product. Every $\mathfrak{k}_r \cong \mathfrak{su}(q_r)$ acts nontrivially because the embedding in $\mathfrak{su}(N)$ is faithful. Its least nontrivial complex representation has dimension q_r [5]. Thus $d_r \geq q_r$ for every r. The equality $\prod_r d_r = N = \prod_r q_r$ forces $d_r = q_r$ factor by factor.

Each W_r is equivalent either to the standard q_r-dimensional representation of $\mathfrak{su}(q_r)$ or to its dual. In either case, the image of $\mathfrak{k}_r$ generates $\operatorname{End}(W_r)$ as a unital complex algebra. The tensor product above therefore supplies a unitary factorization of type $\boldsymbol{q}$ whose one-site traceless sector is $-i\mathfrak{k}$. Since $\mathcal{X} = -i\mathfrak{m} = (-i\mathfrak{k})^{\perp}$, Eq. (2) identifies $\mathcal{X}$ with the joint one-site marginal kernel. This proves statement (a).

The decomposition of a semisimple Lie algebra into simple ideals is unique up to permutation. Let $\operatorname{Alg}_{\mathbb{C}}(\mathfrak{k}_r, \mathbf{1})$ denote the smallest unital complex associative subalgebra of $\operatorname{End}(\mathcal{H})$ containing the represented ideal $\mathfrak{k}_r$. Each recovered ideal generates

$$\mathcal{A}_r = \operatorname{Alg}_{\mathbb{C}}(\mathfrak{k}_r, \mathbf{1}) = \iota_r\big(\operatorname{End}(\mathcal{H}_r)\big).$$

Thus every simple ideal recovers one commuting full matrix algebra. The algebras $\mathcal{A}_r$ commute, and multiplication gives

$$\bigotimes_{r=1}^{m} \mathcal{A}_r \cong \operatorname{End}(\mathcal{H}).$$

Only local changes of basis and permutations among equal-dimensional simple ideals remain. □

4 Marginals and correlation fibres

4.1 Linear marginal quotient

Fix a factorization recovered by Theorem 3.1 and write $\mathcal{C} = \mathcal{X}$. Define the linear marginal map by

$$\operatorname{Marg} : \operatorname{Herm}_0(\mathcal{H}) \longrightarrow \bigoplus_{r=1}^{m} \operatorname{Herm}_0(\mathcal{H}_r),$$
$$A \longmapsto \big(\operatorname{Tr}_{\bar{1}} A, \ldots, \operatorname{Tr}_{\bar{m}} A\big).$$

Its kernel is $\mathcal{C}$. The map is onto. To see this, put traceless $A_r \in \operatorname{Herm}_0(\mathcal{H}_r)$ and form the global operator

$$B = \sum_{r=1}^{m} \frac{q_r}{N} A_r^{(r)}.$$

The rth summand has rth marginal A_r. Its other one-site marginals vanish because A_r is traceless. Hence $\operatorname{Marg}(B) = (A_1, \ldots, A_m)$. The resulting exact sequence is

$$0 \longrightarrow \mathcal{C} \longrightarrow \operatorname{Herm}_0(\mathcal{H}) \xrightarrow{\operatorname{Marg}} \bigoplus_{r=1}^{m} \operatorname{Herm}_0(\mathcal{H}_r) \longrightarrow 0 \tag{4}$$

Two traceless global operators have the same one-site data exactly when their difference lies in $\mathcal{C}$. Thus the exact sequence says in one line that the marginal map is onto and loses precisely the correlation kernel.

The dimension of this lost sector can be read either from the exact sequence or from the support decomposition:

$$\begin{aligned}\dim_{\mathbb{R}} \mathcal{C} &= N^2 - 1 - \sum_{r=1}^{m} (q_r^2 - 1) \\ &= \sum_{\substack{S \subseteq [m] \\ |S| \geq 2}} \prod_{r \in S} (q_r^2 - 1).\end{aligned} \tag{5}$$

It is positive for every $m \geq 2$ and $q_r \geq 2$.

4.2 Equal-marginal states

Let $\mathcal{D}(\mathcal{H})$ be the density-operator space. The same partial-trace formula defines the affine state map

$$\begin{aligned}\mathrm{Marg}_{\mathrm{st}} : \mathcal{D}(\mathcal{H}) &\longrightarrow \prod_{r=1}^{m} \mathcal{D}(\mathcal{H}_r), \\ \rho &\longmapsto (\mathrm{Tr}_{\bar{1}}\, \rho, \ldots, \mathrm{Tr}_{\bar{m}}\, \rho).\end{aligned}$$

It is onto because the tensor product of any tuple of one-site states realizes that tuple. The fibre through ρ is

$$\mathrm{Marg}_{\mathrm{st}}^{-1}\big(\mathrm{Marg}_{\mathrm{st}}(\rho)\big) = \big(\rho + \mathcal{C}\big) \cap \mathcal{D}(\mathcal{H}). \tag{6}$$

Positive definiteness is open in the trace-one affine space. Hence the fibre through every full-rank state is locally an affine manifold with direction space $\mathcal{C}$ and dimension given by Eq. (5).

The two-qubit case gives an explicit witness. For $0 < \epsilon < 1/4$, the states

$$\rho_{\pm} = \frac{\mathbf{1}}{4} \pm \epsilon\, \sigma_3 \otimes \sigma_3$$

are positive definite and distinct. Both have reduced state $\mathbf{1}/2$ at each site. Their difference lies entirely in the nine-dimensional correlation kernel.

4.3 Local state insufficiency

The embedding ι_r defined in Section 3.1 turns a local observable into a global operator. A state ρ defines the global functional $\varphi_\rho(A) = \mathrm{Tr}(\rho A)$ and the one-site restrictions $\varphi_r = \varphi_\rho \circ \iota_r$. A factorization is locally state-sufficient only if $\rho \mapsto (\varphi_1, \ldots, \varphi_m)$ is injective. No nontrivial recognized factorization has this property. The tuple of restrictions fixes the component of ρ in $\mathbb{R}\mathbf{1} \oplus \mathcal{L}_{\mathrm{loc}}$ and leaves its $\mathcal{C}$ component undetermined. The missing state information is exactly the correlation kernel reconstructed in Theorem 3.1.

5 Cut-relative separability

Let $\mathbb{P}(\mathcal{H})$ denote the projective Hilbert space, whose points are one-dimensional complex subspaces of $\mathcal{H}$. A nonzero vector ψ defines the ray $[\psi] \in \mathbb{P}(\mathcal{H})$. This distinction removes normalization and global phase from the pure-state description. A *cut* means an ordered tensor-product factorization of $\mathcal{H}$ with the fixed factor dimensions $\boldsymbol{q} = (q_1, \ldots, q_m)$.

5.1 Product states at a fixed cut

A density operator is fully product relative to the recovered factorization when $\rho = \bigotimes_r \rho_r$. Equivalently, its global functional is multiplicative across the local embeddings:

$$\varphi_\rho\left(\prod_{r=1}^{m} \iota_r(A_r)\right) = \prod_{r=1}^{m} \varphi_r(A_r) \tag{7}$$

for all $A_r \in \mathrm{End}(\mathcal{H}_r)$. Elementary tensor operators span $\mathrm{End}(\mathcal{H})$, so equality of the two functionals on these operators forces $\rho = \bigotimes_r \rho_r$. Productness is therefore defined only after the embeddings have been fixed.

5.2 Product cuts for a fixed ray

The projective local-unitary group of the reference factorization is

$$\begin{aligned}&K := \{[U_1 \otimes \cdots \otimes U_m] : U_r \in \mathrm{U}(q_r)\} \subseteq G, \\ &K \cong \prod_{r=1}^{m} \mathrm{PU}(q_r), \qquad \mathrm{TPS}_{\boldsymbol{q}}(\mathcal{H}) := G/K.\end{aligned} \tag{8}$$

The group K changes local frames without changing the ordered factorization. We use $\mathrm{TPS}_{\boldsymbol{q}}(\mathcal{H})$ as a descriptive symbol for the space of ordered tensor-product structures of type $\boldsymbol{q}$. Let e be the identity of G and let $o = eK$ be the reference cut. A general cut is denoted by $\tau = gK$. It is obtained by conjugating the reference factor algebras by g. Permuting equal-dimensional factors can produce a different ordered point with the same unlabeled kernel. Such permutations preserve full productness, so they do not affect the dimension or measure statements below.

Let Σ_o be the fully product pure-ray locus of the reference cut and write

$$\Sigma_o = \left\{ \left[\bigotimes_{r=1}^{m} \psi_r \right] : 0 \neq \psi_r \in \mathcal{H}_r \right\}.$$

For a general cut $\tau = gK$, define

$$\Sigma_\tau = g\Sigma_o.$$

The notation $g\Sigma_o$ uses the projective action induced by any unitary representative of g. The set Σ_τ contains exactly the pure rays that are fully product relative to the cut τ.

For a fixed ray $[\psi] \in \mathbb{P}(\mathcal{H})$, its product-cut locus is

$$\mathcal{P}_{[\psi]} = \{\tau \in \mathrm{TPS}_{\boldsymbol{q}}(\mathcal{H}) : [\psi] \in \Sigma_\tau\}. \tag{9}$$

Choose a reference product ray $[\phi]$ and $u \in G$ with $u[\phi] = [\psi]$. Define the ray stabilizer by

$$G_{[\psi]} := \{g \in G : g[\psi] = [\psi]\}.$$

Each local projective unitary group acts transitively on its local pure rays, so K acts transitively on Σ_o. A coset gK belongs to $\mathcal{P}_{[\psi]}$ exactly when g differs from u by a ray stabilizer on the left and a local frame transformation on the right. Therefore

$$\mathcal{P}_{[\psi]} = G_{[\psi]} uK/K \cong G_{[\psi]}/(G_{[\psi]} \cap uKu^{-1}).$$

The orbit of a ray is $\mathbb{CP}^{N-1}$, with real dimension $2(N-1)$. Since $\dim G = N^2 - 1$, its stabilizer has dimension

$$\dim G_{[\psi]} = (N^2 - 1) - 2(N-1) = (N-1)^2.$$

At a product ray, the stabilizer inside the rth local factor has dimension $(q_r - 1)^2$. Hence

$$\dim(G_{[\psi]} \cap uKu^{-1}) = \sum_{r=1}^{m} (q_r - 1)^2.$$

Subtracting these dimensions from $\dim(G/K) = N^2 - 1 - \sum_r (q_r^2 - 1)$ gives

$$\operatorname{codim}_{\mathrm{TPS}_{\boldsymbol{q}}(\mathcal{H})} \mathcal{P}_{[\psi]} = 2\left[N - 1 - \sum_{r=1}^{m} (q_r - 1)\right] > 0. \tag{10}$$

The inequality follows by expanding $N = \prod_r (1 + (q_r - 1))$. The compact orbit $\mathcal{P}_{[\psi]}$ is therefore a closed embedded submanifold of positive codimension. It has zero measure for the normalized G-invariant measure on factorization space.

Transitivity of G maps a reference product ray to any chosen ray. It also maps a reference nonproduct ray to that same ray. Hence

$$\bigcup_{\tau \in G/K} \Sigma_\tau = \mathbb{P}(\mathcal{H}), \qquad \bigcap_{\tau \in G/K} \Sigma_\tau = \varnothing. \tag{11}$$

Every pure ray is product for some cut, no pure ray is product for every cut, and almost every ordered cut makes any fixed ray nonproduct. Full pure-state separability is therefore a relation between a ray and a factorization. It is not a nonconstant G-invariant unary property of the ray.

The factor algebras remain distinct commuting subalgebras. Standard mixed-state separability remains the convex hull of cut-relative product states. These results neither prove nor invalidate Bell local causality and predict no Bell violation.

6 Correlation geometry

6.1 Tangent directions

For the reference cut, multiplication by i gives the orthogonal reductive splitting

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}, \qquad \mathfrak{k} = i\mathcal{L}_{\mathrm{loc}}, \qquad \mathfrak{m} = i\mathcal{C}. \tag{12}$$

The vertical directions $\mathfrak{k}$ change local frames while leaving the cut fixed. The complementary directions $\mathfrak{m}$ change the cut itself. The letter $\mathfrak{m}$ is the conventional notation for a reductive complement on a general homogeneous space. No symmetric-space relation $[\mathfrak{m}, \mathfrak{m}] \subseteq \mathfrak{k}$ is assumed.

At the cut $\tau = gK$, the correlation sector is $\mathcal{C}_\tau = \mathrm{Ad}_g \mathcal{C}$. This definition is independent of the representative g because K preserves the reference kernel. The associated bundle $G \times_K \mathcal{C}$ records a cut together with a correlation operator expressed in a local frame. Its equivalence relation is $[gk, C] = [g, \mathrm{Ad}_k C]$ for $k \in K$ and $C \in \mathcal{C}$.

Equip G/K with the normal homogeneous metric induced by $-\operatorname{Tr}(M_1 M_2)$ on $\mathfrak{m}$, and transport the Hilbert–Schmidt metric on $\mathcal{C}$ to every fibre of the associated bundle. Write $T\,\mathrm{TPS}_{\boldsymbol{q}}(\mathcal{H})$ for the full tangent bundle of factorization space. With these metrics, the map

$$\begin{aligned} G \times_K \mathcal{C} &\longrightarrow T\,\mathrm{TPS}_{\boldsymbol{q}}(\mathcal{H}), \\ [g, C] &\longmapsto \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} g \exp(tiC) K \end{aligned} \tag{13}$$

is a G-equivariant isometric vector-bundle isomorphism. The tangent space at the reference cut is $\mathfrak{g}/\mathfrak{k}$. The splitting in Eq. (12) identifies this quotient with $\mathfrak{m} = i\mathcal{C}$, and left translation gives every other fibre. Thus correlation operators represent infinitesimal changes of factorization, not additional local observables.

6.2 Kernel stabilizer

The kernel also determines its local symmetry group. For a subspace W, its stabilizer under conjugation is $\mathrm{Stab}_G(W) = \{g \in G : \mathrm{Ad}_g W = W\}$. The group normalizer of K is $N_G(K) = \{g \in G : gKg^{-1} = K\}$. Let S_m be the symmetric group on the m factor labels, and define

$$\Pi_{\boldsymbol{q}} = \{\sigma \in S_m : q_{\sigma(r)} = q_r \text{ for every } r\}.$$

A superscript $\circ$ denotes the connected component containing the identity. The stabilizer and its identity component satisfy

$$\begin{aligned} \mathrm{Stab}_G(\mathcal{C}) &= \mathrm{Stab}_G(\mathfrak{m}) = N_G(K), \\ \mathrm{Stab}_G(\mathfrak{m})^\circ &= K. \end{aligned} \tag{14}$$

Conjugation preserves the Hilbert–Schmidt inner product. Hence an element of G preserves $\mathfrak{m}$ exactly when it preserves the orthogonal complement $\mathfrak{k}$. Preserving $\mathfrak{k}$ is equivalent to normalizing the connected subgroup K with Lie algebra $\mathfrak{k}$. A normalizer element permutes the simple ideals of $\mathfrak{k}$. Their dimensions allow only permutations among equal-dimensional factors. After such a permutation, every remaining automorphism of a local full matrix algebra is inner. Therefore

$$1 \longrightarrow K \longrightarrow \mathrm{Stab}_G(\mathfrak{m}) \longrightarrow \Pi_{\boldsymbol{q}} \longrightarrow 1.$$

The first arrow includes the local group, and the last map records the permutation induced on equal-dimensional factors. Exactness says that the elements inducing no permutation are precisely the local transformations. The sequence splits after unitary identifications between equal-dimensional factor spaces have been chosen. The orbit of the unlabeled kernel is $G/N_G(K)$, while G/K retains the ordering of equal-dimensional factors.

6.3 Invariant connection

An invariant connection on $G \to G/K$ is determined by an Ad_K-invariant horizontal complement of $\mathfrak{k}$ at the identity. Every complement is the graph over $\mathfrak{m}$ of a linear map $\Gamma : \mathfrak{m} \to \mathfrak{k}$. This means that every vector in the complement has a unique form $M + \Gamma(M)$ with $M \in \mathfrak{m}$. Invariance requires Γ to intertwine the K actions. Consider a support sector $iV_S \subseteq \mathfrak{m}$ and a one-site ideal $\mathfrak{k}_r \subseteq \mathfrak{k}$. If $r \in S$, choose $s \in S \setminus \{r\}$. If $r \notin S$, choose any $s \in S$. The factor $K_s \cong \mathrm{PU}(q_s)$ is the sth factor of K. It acts through its adjoint representation on V_S and trivially on $\mathfrak{k}_r$. Its adjoint representation has no invariant covector, so every K-equivariant map $iV_S \to \mathfrak{k}_r$ is zero. Summing over the support sectors and one-site ideals gives the following result. Here $\mathrm{Hom}_K(\mathfrak{m}, \mathfrak{k})$ denotes the real vector space of K-equivariant linear maps from $\mathfrak{m}$ to $\mathfrak{k}$.

$$\mathrm{Hom}_K(\mathfrak{m}, \mathfrak{k}) = 0.$$

The absence of an invariant covector means that the adjoint representation has no nonzero invariant real linear functional. The complement $\mathfrak{m}$ is the unique Ad_K-invariant complement.

By Wang's classification, invariant principal connections on $G \to G/K$ correspond to equivariant projections onto $\mathfrak{k}$ [17]. The vanishing intertwiner space leaves one such projection. Let $\mathrm{pr}_{\mathfrak{k}} : \mathfrak{g} \to \mathfrak{k}$ be the Hilbert–Schmidt orthogonal projection defined by Eq. (12). For $v \in T_g G$, the left Maurer–Cartan form sends v to $(\mathrm{d}L_{g^{-1}})_g v \in \mathfrak{g}$. The connection and its horizontal space are

$$\begin{aligned} \omega_g(v) &= \mathrm{pr}_{\mathfrak{k}}\big((\mathrm{d}L_{g^{-1}})_g v\big), \\ \mathrm{Hor}_g &= (\mathrm{d}L_g)_e \mathfrak{m}. \end{aligned} \tag{15}$$

Here $L_g(h) = gh$ is left multiplication and $(\mathrm{d}L_g)_e$ is its differential at the identity. This is the only principal K-connection invariant under G. A curve $g(t)$ is horizontal exactly when $g^{-1}\dot{g} \in \mathfrak{m}$, where this expression denotes left trivialization. Equivalently, under right trivialization, $\dot{g}g^{-1} \in \mathrm{Ad}_g \mathfrak{m} = i\mathcal{C}_\tau$ at the cut $\tau = gK$.

6.4 Curvature and local holonomy

Let Ω denote the curvature two-form of ω. The Maurer–Cartan equation gives its value at the identity $e \in G$:

$$\Omega_e(M_1, M_2) = -\,\mathrm{pr}_{\mathfrak{k}}[M_1, M_2] \quad \text{for } M_1, M_2 \in \mathfrak{m}. \tag{16}$$

The value $\Omega_e(M_1, M_2)$ is the local-frame component produced by commuting the horizontal directions M_1 and M_2. The relevant curvature values come from pairs of two-site correlation directions. Fix a site r, choose $s \neq r$, and let $\{D_\alpha\}_{\alpha=1}^{q_s^2-1}$ be a Hilbert–Schmidt orthonormal basis of $\mathrm{Herm}_0(\mathcal{H}_s)$. Its quadratic Casimir identity is

$$\sum_\alpha D_\alpha^2 = \frac{q_s^2 - 1}{q_s} \mathbf{1}_s.$$

The sum commutes with every unitary on $\mathcal{H}_s$, so it is scalar. Taking its trace fixes the displayed coefficient.

Choose $A, B \in \mathrm{Herm}_0(\mathcal{H}_r)$ and define two correlation directions for each basis element:

$$P_\alpha = iA^{(r)} D_\alpha^{(s)}, \qquad Q_\alpha = iB^{(r)} D_\alpha^{(s)}.$$

Both lie in $\mathfrak{m}$. Summing their commutators removes the helper site s and leaves a generator acting only at site r:

$$\sum_\alpha [P_\alpha, Q_\alpha] = -\frac{q_s^2 - 1}{q_s} [A, B]^{(r)}. \tag{17}$$

Commutators of traceless Hermitian matrices span $\mathfrak{su}(q_r)$. Varying r in Eq. (17) proves

$$\mathrm{span}_{\mathbb{R}}\{\Omega_e(M_1, M_2) : M_1, M_2 \in \mathfrak{m}\} = \mathfrak{k}. \tag{18}$$

Equation (18) states explicitly that real linear combinations of curvature values exhaust the local Lie algebra. The connection is therefore nonflat, and its curvature spans every local projective generator.

The Ambrose–Singer theorem identifies the restricted holonomy algebra with the algebra generated by curvature values along horizontal lifts [2]. Equation (18) spans all of $\mathfrak{k}$ at the base point. Write $\mathrm{Hol}_e(\omega)$ for the holonomy group based at the point e in the fibre over the reference cut $o = eK$. Since K is connected,

$$\mathrm{Hol}_e(\omega) = K. \tag{19}$$

The chosen point identifies the holonomy group as a concrete subgroup of K. Every $k \in K$ is the endpoint of the horizontal lift, starting at e, of a contractible piecewise-smooth loop in factorization space.

6.5 Projective Lie closure

Let $[\mathfrak{m},\mathfrak{m}]_{\mathbb{R}}$ denote the real linear span of all commutators $[M_1, M_2]$ with $M_1, M_2 \in \mathfrak{m}$. Let $\mathrm{Lie}_{\mathbb{R}}(\mathfrak{m})$ denote the smallest real Lie subalgebra of $\mathfrak{g}$ containing $\mathfrak{m}$. Equation (17) shows that $\mathfrak{k} \subseteq [\mathfrak{m},\mathfrak{m}]_{\mathbb{R}}$. Combining this inclusion with the reductive splitting gives

$$\mathfrak{g} = \mathfrak{m} + [\mathfrak{m},\mathfrak{m}]_{\mathbb{R}}, \qquad \mathrm{Lie}_{\mathbb{R}}(\mathfrak{m}) = \mathfrak{su}(N). \tag{20}$$

Thus every projective unitary direction is either a correlation direction or a real linear combination of commutators of two correlation directions. This is the meaning of bracket generation at step two. The first bracket span itself has one exceptional form:

$$[\mathfrak{m},\mathfrak{m}]_{\mathbb{R}} = \begin{cases} \mathfrak{k}, & m = 2 \text{ and } q_1 = q_2 = 2, \\ \mathfrak{g}, & \text{otherwise.} \end{cases} \tag{21}$$

In both cases, the Lie algebra generated by $\mathfrak{m}$ is all of $\mathfrak{su}(N)$.

Let $\mathfrak{u}(N)$ be the real Lie algebra of all skew-Hermitian $N \times N$ matrices. Adding the scalar direction gives

$$\mathfrak{u}(N) = i\mathbb{R}\mathbf{1} \oplus \mathrm{Lie}_{\mathbb{R}}(\mathfrak{m}). \tag{22}$$

Scalar phase is the only unitary Lie-algebra direction outside the closure generated by the correlation kernel.

7 Mathematical scope

The inverse datum is an unlabeled subspace $\mathcal{X} \subseteq \mathrm{Herm}_0(\mathcal{H})$. Equation (3) recognizes whether that datum is a multipartite correlation kernel without using a prior factorization in the converse. It does not select $\mathcal{X}$ from a bare Hilbert space. Equal-dimensional factor labels are not encoded in the subspace.

Equations (10) and (11) exclude intrinsic full pure-state separability. They do not deny product rays relative to a recognized cut. The maximally mixed state is product for every factorization, so the empty-intersection statement does not extend to mixed states.

The unique G-invariant connection is nonflat, and every element of K is its holonomy around a contractible loop. Hence no G-invariant principal connection identifies local frames by endpoints alone. This does not exclude noninvariant connections or define physical transport.

The term correlation kernel is algebraic. For a Hamiltonian $H \in \mathcal{C}$, the one-site marginals of $\dot\rho = -i[H,\rho]$ need not vanish. Equation (20) implies algebraic controllability only for a system with arbitrary signed piecewise-constant generators taking values in $\mathfrak{m}$. One fixed interaction need not have the same Lie closure. The connection supplies no Hamiltonian, intervention class, signaling protocol, propagation law, or spacetime interpretation. Correlation-induced holonomies in other state and control bundles are distinct constructions [8, 12].

A Exact first-bracket span

Two qubits. For operators X_1, X_2 on the first factor and Y_1, Y_2 on the second, write $\{X,Y\} = XY + YX$ for the anticommutator. Then

$$[X_1 \otimes Y_1, X_2 \otimes Y_2] = \frac{1}{2}\big([X_1,X_2] \otimes \{Y_1,Y_2\} + \{X_1,X_2\} \otimes [Y_1,Y_2]\big),$$

The anticommutator of two traceless Hermitian 2×2 matrices is scalar. A commutator of two two-site correlation directions therefore contains only one-site terms. This gives $[\mathfrak{m},\mathfrak{m}]_{\mathbb{R}} \subseteq \mathfrak{k}$. The reverse inclusion follows from Eq. (17), so equality holds.

Other bipartite dimensions. For every bipartite type other than 2×2, one factor has dimension $q_s > 2$. On that factor choose

$$D = \mathrm{diag}(1, -1, 0, \ldots, 0).$$

This operator is traceless and its square is not scalar. Define

$$c = \frac{\mathrm{Tr}(D^2)}{q_s} = \frac{2}{q_s}, \qquad D_0 = D^2 - c\mathbf{1}_s.$$

Then $D_0 \neq 0$ and $\mathrm{Tr}\, D_0 = 0$. Let r label the other factor and choose noncommuting $A, B \in \mathrm{Herm}_0(\mathcal{H}_r)$. The two correlation directions

$$P = iA^{(r)}D^{(s)}, \qquad Q = iB^{(r)}D^{(s)}$$

satisfy

$$[P,Q] = -c[A,B]^{(r)} - [A,B]^{(r)}D_0^{(s)}.$$

The first term is local and already lies in the first-bracket span by Eq. (17). Subtracting it leaves the nonzero correlation term $-[A,B]^{(r)}D_0^{(s)}$.

In the bipartite case, $\mathfrak{m}$ is the external tensor product of the two real adjoint representations on $\mathrm{Herm}_0(\mathcal{H}_1)$ and $\mathrm{Herm}_0(\mathcal{H}_2)$. Each adjoint representation is irreducible, and their external tensor product is irreducible under K. The bracket span is K-invariant. Once it contains one nonzero correlation direction, it therefore contains all of $\mathfrak{m}$. Together with its local part, it is all of $\mathfrak{g}$.

Three or more factors. Let $m \geq 3$. Identity factors are suppressed in the site-embedded operator products. Fix a support set S with $|S| \geq 3$ and choose distinct sites $r, s \in S$. Choose $A \in \mathrm{Herm}_0(\mathcal{H}_r)$ and $B, C \in \mathrm{Herm}_0(\mathcal{H}_s)$. For every $u \in S \setminus \{r,s\}$, choose $D_u \in \mathrm{Herm}_0(\mathcal{H}_u)$. Define

$$P = iA^{(r)}B^{(s)}, \qquad Q = iC^{(s)} \prod_{u \in S\setminus\{r,s\}} D_u^{(u)}.$$

Both P and Q have support on at least two sites, so they belong to $\mathfrak{m}$. Their commutator is

$$[P,Q] = -A^{(r)}[B,C]^{(s)} \prod_{u\in S\setminus\{r,s\}} D_u^{(u)}.$$

Commutators of B and C span the traceless skew-Hermitian operators on site s. Varying all displayed operators therefore spans every sector iV_S with $|S| \geq 3$.

It remains to obtain a sector supported on two sites r and s. Choose a third site t. Let $\{D_\alpha\}_{\alpha=1}^{q_t^2-1}$ be a Hilbert–Schmidt orthonormal basis of $\mathrm{Herm}_0(\mathcal{H}_t)$, and take $A, B \in \mathrm{Herm}_0(\mathcal{H}_r)$ and $C \in \mathrm{Herm}_0(\mathcal{H}_s)$. Define

$$P_\alpha = iA^{(r)}D_\alpha^{(t)}, \qquad Q_\alpha = iB^{(r)}C^{(s)}D_\alpha^{(t)}.$$

Both are correlation directions. The Casimir identity on site t gives

$$\sum_\alpha [P_\alpha, Q_\alpha] = -\frac{q_t^2-1}{q_t}[A,B]^{(r)}C^{(s)}.$$

These brackets span every support-two sector. Hence $\mathfrak{m} \subseteq [\mathfrak{m},\mathfrak{m}]_{\mathbb{R}}$. Combining this with $\mathfrak{k} \subseteq [\mathfrak{m},\mathfrak{m}]_{\mathbb{R}}$ from Eq. (17) proves Eq. (21) for every multipartite type.

Author contributions

The author conceived the work, formulated and verified the theorem, and assumes full responsibility. Large language models assisted literature discovery, adversarial proof checking, LaTeX drafting, and language editing.

Data and code availability

No data or computational results were used.

Competing interests

The author declares no competing interests.

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