Homogeneous Geometry of Product Cuts for Fixed Quantum States

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Fix a density operator and vary its ordered tensor-product structure at fixed factor dimensions. We classify all cuts at which the state is exactly product. Compatible Kronecker factorizations of the spectrum label finitely many disjoint compact homogeneous components. Global and local eigenvalue multiplicities determine the exact dimension and codimension of every component. Every nonempty component for a state other than I/N obeys the sharp lower bound $2[N-1-\sum_r(q_r-1)]$. Pure states and only pure states attain equality. Every component for a compatible mixed state other than I/N has codimension at least two larger. The smallest component codimension is also the leading exponent in the small-radius volume of near-product cuts. Thus exact product decomposition of a fixed state is exceptional even when its spectrum admits one. The classification concerns exact productness, not mixed-state separability.

1 Introduction

A density operator is product only relative to a specified subsystem structure. On an abstract Hilbert space, a tensor-product structure supplies the factor algebras, partial traces, local unitary group, and the predicate used to test productness. Different structures on the same Hilbert space can therefore classify the same operator differently [1–5].

The unitary-orbit formulation keeps one reference factorization fixed and conjugates the state globally. This is mathematically equivalent to fixing the state and varying the factorization. Previous work treats correlation extrema and subsystem-entropy averages on global unitary orbits [6, 7]. Modi and Gu identify the product-compatible spectra and show that they have dimension $\sum_r(q_r-1)$ inside the $(N-1)$-dimensional spectral simplex [8].

Jing classifies local-unitary classes on a fixed global spectral orbit by double cosets of the local group and the global spectral stabilizer [9]. Homogeneous geometry is also used for product-state sets and for local-unitary orbits of locally diagonalizable states [10, 11]. Tensor-product structures carry a geometry defined through their local operator spaces [12]. A distinct dynamical problem asks for one factorization that keeps an entire pure-state trajectory product [13].

These results leave a fixed-state inverse problem. Given one density operator, what is the complete subset of tensor-product structures that makes it product, what are the dimensions of its components, and how exceptional are nearby product cuts? The answer depends on both global spectral degeneracies and the degeneracies of every compatible local spectral factorization. Accidental equalities between products of local eigenvalues must be retained. Ignoring them gives incorrect dimensions even for two qubits.

The product-cut locus decomposes into finitely many homogeneous components. For every state other than I/N, every nonempty product-cut component obeys the universal sharp lower bound. Pure states saturate that bound, so their product-cut components have the largest possible dimension among states other than I/N. Every compatible mixed state other than I/N has strictly smaller components. The minimum component codimension controls the leading small-tube volume.

Jing's double cosets describe the ambient local-unitary classes. Here the state is fixed and the variable is a cut in G/K. Product compatibility selects particular centralizer orbits in this cut space. Their stabilizers retain the multiplicities of every compatible local factor. To our knowledge, prior work does not determine this complete multiplicity-resolved product fibre, its exact component codimensions, the sharp pure-state lower envelope, or the associated small-tube exponent.

Product means an exact tensor factorization of the density operator. Mixed state separability means a convex combination of product operators. The two conditions are different.

2 Tensor-product cuts

2.1 Fixed factor type

Fix an integer $m \geq 2$ and factor dimensions $q_r \geq 2$ for $r = 1, \ldots, m$. Set

$$\mathcal{H} = \bigotimes_{r=1}^{m} \mathbb{C}^{q_r}, \qquad N = \prod_{r=1}^{m} q_r. \tag{1}$$

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The ordering labels the factors. Let $G = \mathrm{U}(N)$. The local unitary subgroup is the image

$$K = \{U_1 \otimes \cdots \otimes U_m : U_r \in \mathrm{U}(q_r)\} \subseteq G. \tag{2}$$

The tensor-product map from $\prod_r \mathrm{U}(q_r)$ to K has an $(m-1)$-dimensional phase kernel. Consequently

$$\dim K = \sum_{r=1}^{m} q_r^2 - (m-1). \tag{3}$$

A unitary $U \in G$ transports the reference factor algebras to a new factorization of the same Hilbert space. Right multiplication by a local unitary changes only the bases inside those factors. It does not change the factorization. Equal-dimensional factor permutations are not quotiented. The ordered cut space is therefore

$$\mathcal{T}_{\boldsymbol{q}} = G/K, \qquad \dim \mathcal{T}_{\boldsymbol{q}} = N^2 - \sum_{r=1}^{m} q_r^2 + m - 1. \tag{4}$$

Global phase already lies in K, so using $\mathrm{U}(N)$ or its projective quotient gives the same cut space.

The real Hilbert–Schmidt form is $\langle X, Y\rangle = \mathrm{Re}\,\mathrm{Tr}(X^\dagger Y)$. At the identity coset, it is restricted to the skew-Hermitian operators orthogonal to the Lie algebra of K. Left translation defines the normal homogeneous Riemannian metric at every cut. We write d for its geodesic distance and Vol for its Riemannian volume. The normalized invariant measure is

$$\mu(E) = \frac{\mathrm{Vol}(E)}{\mathrm{Vol}(\mathcal{T}_{\boldsymbol{q}})}. \tag{5}$$

All distances and submanifold volumes use this metric. The measure describes the mathematical space of cuts. It is not a distribution of physically realized subsystem structures.

2.2 One fixed state

Fix a density operator ρ on $\mathcal{H}$. At the cut represented by UK, the same operator has reference coordinates

$$\rho_U = U^\dagger \rho U. \tag{6}$$

Replacing U by Uk with $k \in K$ conjugates the local factors and does not change whether the operator is product. The cut is a fully product cut when there are density operators σ_r on $\mathbb{C}^{q_r}$ such that

$$\rho_U = \sigma_1 \otimes \cdots \otimes \sigma_m. \tag{7}$$

The set of all such cuts is written $\mathcal{P}_\rho \subseteq \mathcal{T}_{\boldsymbol{q}}$.

3 Compatible spectral branches

List the N eigenvalues of ρ with multiplicity. Place this list into an m-dimensional array with side lengths $q_1, \ldots, q_m$. An arrangement is product-compatible when there are probability vectors $p^{(r)} = (p_1^{(r)}, \ldots, p_{q_r}^{(r)})$ satisfying

$$\lambda_{i_1 \cdots i_m} = p_{i_1}^{(1)} \cdots p_{i_m}^{(m)} \tag{8}$$

at every array position. Permuting entries within a local vector changes only a local eigenbasis. Arrangements that give the same ordered tuple of local spectral multisets define one compatible spectral branch.

There are finitely many branches. There are only finitely many assignments of the global eigenvalue occurrences to the array positions. For a fixed assignment that satisfies Eq. (8), sum the array over every index except i_r. Every other probability vector has sum one, so the result is exactly $p_{i_r}^{(r)}$. The arrangement therefore determines every local spectrum uniquely. This argument includes zero eigenvalues and repeated eigenvalues.

Let $\mathcal{A}_\rho$ denote the finite set of compatible branches. For a branch $\alpha \in \mathcal{A}_\rho$, write the distinct eigenvalues of its rth local spectrum as a_{rj}^α. Let n_{rj}^α be their multiplicities, so

$$\sum_j n_{rj}^\alpha = q_r. \tag{9}$$

Choose diagonal local density operators

$$D_{r,\alpha} = \bigoplus_j a_{rj}^\alpha I_{n_{rj}^\alpha}, \qquad D_\alpha = \bigotimes_{r=1}^{m} D_{r,\alpha}. \tag{10}$$

By compatibility, D_α and ρ have the same eigenvalues with the same multiplicities. Choose a unitary V_α such that

$$V_\alpha^\dagger \rho V_\alpha = D_\alpha. \tag{11}$$

Finally, let

$$L_\alpha = \{W \in \mathrm{U}(N) : W D_\alpha = D_\alpha W\} \tag{12}$$

be the unitary centralizer of D_α. It consists of arbitrary unitary rotations inside the eigenspaces of the global product spectrum.

4 Exact geometry of the product-cut locus

Let the distinct eigenvalues of ρ, including zero when present, have multiplicities M_λ. Let ω_c denote the Euclidean volume of the unit ball in $\mathbb{R}^c$.

Theorem 1. For every density operator ρ and ordered factor type $\boldsymbol{q} = (q_1, \ldots, q_m)$, the following three statements hold.

(i) The fully product-cut locus is the finite disjoint union

$$\begin{aligned} \mathcal{P}_\rho &= \bigsqcup_{\alpha \in \mathcal{A}_\rho} \mathcal{M}_\alpha, \\ \mathcal{M}_\alpha &= V_\alpha L_\alpha K/K, \\ \mathcal{M}_\alpha &\cong L_\alpha/(L_\alpha \cap K). \end{aligned} \tag{13}$$

Each $\mathcal{M}_\alpha$ is a compact connected embedded real-analytic submanifold of $\mathcal{T}_{\boldsymbol{q}}$. Its dimension and codimension are

$$\dim \mathcal{M}_\alpha = \sum_\lambda M_\lambda^2 - \sum_{r,j} (n_{rj}^\alpha)^2 + m - 1, \quad (14)$$

$$c_\alpha = N^2 - \sum_\lambda M_\lambda^2 - \sum_r q_r^2 + \sum_{r,j} (n_{rj}^\alpha)^2. \quad (15)$$

Equivalently, c_α is the dimension of the global unitary orbit of ρ minus the sum of the unitary-orbit dimensions of its local factors.

(ii) If $\rho = I/N$, then $\mathcal{P}_\rho = \mathcal{T}_{\boldsymbol{q}}$. If $\rho \neq I/N$, every nonempty component has even positive codimension and

$$c_\alpha \geq c_{\mathrm{pure}}(\boldsymbol{q}) := 2\left[N - 1 - \sum_{r=1}^m (q_r - 1)\right]. \quad (16)$$

Equality holds exactly when ρ is pure. Every compatible mixed state other than I/N satisfies $c_\alpha \geq c_{\mathrm{pure}}(\boldsymbol{q}) + 2$.

(iii) Assume $\rho \neq I/N$ and $\mathcal{A}_\rho$ is nonempty. Set $c_* = \min_\alpha c_\alpha$. For sufficiently small $\varepsilon > 0$,

$$\mu\{x \in \mathcal{T}_{\boldsymbol{q}} : d(x, \mathcal{P}_\rho) < \varepsilon\} = \frac{\omega_{c_*}}{\mathrm{Vol}(\mathcal{T}_{\boldsymbol{q}})} \left(\sum_{\alpha: c_\alpha = c_*} \mathrm{Vol}(\mathcal{M}_\alpha)\right) \varepsilon^{c_*} + o(\varepsilon^{c_*}). \quad (17)$$

4.1 Completeness of the components

Suppose UK is a product cut. The spectra of the factors in Eq. (7) define one branch α. Local eigenbasis changes give a $k \in K$ such that

$$U^\dagger \rho U = k D_\alpha k^\dagger. \quad (18)$$

Both Uk and V_α now pull ρ back to D_α. It follows that $W = V_\alpha^\dagger U k$ commutes with D_α. Hence $UK = V_\alpha W K$ with $W \in L_\alpha$. If $W \in L_\alpha$, then $(V_\alpha W)^\dagger \rho (V_\alpha W) = D_\alpha$, so every cut in the displayed orbit is product. This proves Eq. (13).

At a fixed cut, the factors of a normalized product state are its one-site partial traces. Their ordered local spectra are therefore unique. Components belonging to distinct branches cannot intersect. Changing the aligning unitary V_α only reparametrizes the same component.

The centralizer L_α is a product of unitary groups acting on the global eigenspaces. Therefore

$$\dim L_\alpha = \sum_\lambda M_\lambda^2. \quad (19)$$

A local unitary stabilizes D_α exactly when it stabilizes every local factor $D_{r,\alpha}$. If every U_r stabilizes its local factor, their tensor product stabilizes D_α. Conversely, partial tracing the stabilized product over every factor except r gives $U_r D_{r,\alpha} U_r^\dagger = D_{r,\alpha}$, because all other local factors have trace one. Accounting for the $(m-1)$-dimensional phase kernel in the tensor-product map gives

$$\dim(L_\alpha \cap K) = \sum_{r,j} (n_{rj}^\alpha)^2 - (m-1). \quad (20)$$

The orbit-stabilizer dimension formula gives Eq. (14). Subtracting it from Eq. (4) gives Eq. (15). Compactness, connectedness, and real analyticity follow from the corresponding properties of the unitary centralizers and their homogeneous quotients.

4.2 Sharp codimension bound

For a Hermitian operator X on an n-dimensional space, let $\nu(X)$ count the unordered pairs of eigenbasis positions that carry unequal eigenvalues. If its distinct eigenvalues have multiplicities $m_1, m_2, \ldots$, then

$$\nu(X) = \frac{1}{2}\left(n^2 - \sum_j m_j^2\right). \quad (21)$$

The unitary conjugacy orbit of X has real dimension $2\nu(X)$. Applied to a product representative $D_\alpha = \bigotimes_r D_{r,\alpha}$, Eq. (15) becomes

$$\frac{c_\alpha}{2} = \nu(D_\alpha) - \sum_r \nu(D_{r,\alpha}). \quad (22)$$

This also proves that every component codimension is even.

Let ℓ_r be the largest eigenvalue of $D_{r,\alpha}$ and let u_r be its multiplicity. Put $U_0 = \prod_r u_r$. Since each local factor has trace one, $\ell_r > 0$. The largest eigenvalue of the product is $\prod_r \ell_r$, its multiplicity is U_0, and every other product eigenvalue is strictly smaller.

Separate the pairs counted by ν into those meeting the largest-eigenvalue sector and those lying completely below it. Write $\nu_<$ for the second count. Then

$$\nu(D_{r,\alpha}) = u_r(q_r - u_r) + \nu_<(D_{r,\alpha}),$$
$$\nu(D_\alpha) = U_0(N - U_0) + \nu_<(D_\alpha). \quad (23)$$

Every unequal pair below the largest sector of a local factor injects into a different unequal pair below the largest global sector. Fix one largest-sector index in every other factor and vary only the two indices of the selected local pair. The two global eigenvalues acquire the same positive multiplier, so they remain unequal. Images from different factors differ in different coordinates and cannot coincide. Therefore

$$\nu_<(D_\alpha) \geq \sum_r \nu_<(D_{r,\alpha}). \quad (24)$$

Equations (22)–(24) give

$$\frac{c_\alpha}{2} \geq U_0(N-U_0) - \sum_r u_r(q_r-u_r). \tag{25}$$

It remains to bound the largest-sector contribution. For $a, b \geq 2$, $1 \leq u < a$, and $1 \leq v \leq b$, direct expansion gives

$$\begin{aligned} &uv(ab-uv) - u(a-u) - v(b-v) \\ &\quad = u(a-u)(vb-1) + v(b-v)(u^2-1) \\ &\quad \geq (a-1)(b-1). \end{aligned} \tag{26}$$

The inequality uses $u(a-u) \geq a-1$ and $vb-1 \geq b-1$. The second term on the right is nonnegative. Equality in these bounds together with vanishing of the second term occurs only when $u = v = 1$.

Because D_α is not maximally mixed, at least one local factor has $u_r < q_r$. Put that factor first. Define

$$Q_k = \prod_{r=1}^{k} q_r, \qquad U_k = \prod_{r=1}^{k} u_r. \tag{27}$$

At step k, group the first $k-1$ factors into one effective factor. The left side of Eq. (26) is then

$$\begin{aligned} &U_k(Q_k - U_k) - U_{k-1}(Q_{k-1} - U_{k-1}) \\ &\quad - u_k(q_k - u_k). \end{aligned} \tag{28}$$

Summing these increments from $k = 2$ to m cancels every intermediate term and yields

$$U_0(N-U_0) - \sum_r u_r(q_r-u_r) \geq N - 1 - \sum_r (q_r - 1). \tag{29}$$

The increments telescope because

$$N - 1 - \sum_r (q_r - 1) = \sum_{k=2}^{m} (Q_{k-1} - 1)(q_k - 1). \tag{30}$$

Equality at every step forces $u_r = 1$ for all r.

Suppose all largest local eigenvalues are simple but factor r has a positive subleading eigenvalue. Choose the global index with that subleading value in factor r and largest values in every other factor. Choose $s \neq r$ and form a second index by lowering only factor s. The two product eigenvalues are unequal and both lie below the largest global sector. This pair is outside every pair used in Eq. (24). It differs in factor s, but its factor-r coordinate is already nonmaximal, whereas every injected pair fixes all other coordinates at their maxima. The inequality in Eq. (24) is therefore strict.

Equality in Eq. (16) therefore requires every local factor to have one eigenvalue equal to one and all remaining eigenvalues equal to zero. The global state is pure. Conversely, for pure local factors, $\nu(D_\alpha) = N - 1$ and $\nu(D_{r,\alpha}) = q_r - 1$. Substitution into Eq. (22) gives equality in Eq. (16). This proves the sharp bound and its saturation condition. For a compatible mixed state other than I/N, the inequality is strict. Every component codimension is even, so strictness gives $c_\alpha \geq c_{\mathrm{pure}}(\boldsymbol{q}) + 2$.

4.3 Tube volume

The components in Eq. (13) are finite, compact, and disjoint. Distinct components have positive separation. Each has a positive normal injectivity radius. Choose ε below every such radius and, when more than one component is present, below half the minimum distance between distinct components. The component tubes are then disjoint. The leading term of the standard tube formula for a codimension-c closed submanifold is its volume multiplied by $\omega_c \varepsilon^c$ [14, 15]. After division by the volume of the cut space, only components of smallest codimension contribute at leading order. This gives Eq. (17).

5 Two-qubit cases

For two qubits, $\dim \mathcal{T}_{(2,2)} = 9$. Table 1 shows the geometric regimes. The uniform and pure spectra are $(1/4, 1/4, 1/4, 1/4)$ and $(1, 0, 0, 0)$. A distinct product-compatible spectrum is

$$(0.42, 0.28, 0.18, 0.12) = (0.7, 0.3) \otimes (0.6, 0.4). \tag{31}$$

Because the factors are ordered, exchanging its unequal local spectra gives a second branch. Both branch components have the same dimension.

Table 1: Two-qubit product-cut components. The branch count is for ordered factors. Dimensions and codimensions refer to each branch component. The headings $\sum M^2$ and $\sum n^2$ abbreviate $\sum_\lambda M_\lambda^2$ and $\sum_{r,j} (n_{rj}^\alpha)^2$.

Case	branches	$\sum M^2$	$\sum n^2$	$\dim \mathcal{M}$	c
Uniform	1	16	8	9	0
Pure	1	10	4	7	2
Distinct	2	4	4	1	8
Collision	1	6	4	3	6
Incompatible	0	4	–	–	–

The collision spectrum is $(0.49, 0.21, 0.21, 0.09) = (0.7, 0.3) \otimes (0.7, 0.3)$. The repeated value 0.21 enlarges the global centralizer and raises the component dimension from one to three. This is the simplest example in which the global product multiplicities cannot be inferred by treating all local products as distinct.

For four eigenvalues to form a 2×2 product table, one pairing must have equal cross-products. For $(0.4, 0.3, 0.2, 0.1)$, the three comparisons are $0.4 \cdot 0.1 \neq 0.3 \cdot 0.2$, $0.4 \cdot 0.2 \neq 0.3 \cdot 0.1$, and $0.4 \cdot 0.3 \neq 0.2 \cdot 0.1$. Its product-cut locus is empty.

6 Bipartitions and total correlation

Choose a nonempty proper subset $S \subset \{1, \ldots, m\}$. Each unordered bipartition $S|S^c$ is counted once. Set

$$d_S = \prod_{r \in S} q_r, \qquad d_{S^c} = N/d_S. \tag{32}$$

Let K_S be the subgroup of unitaries local to the grouped factors of dimensions d_S and d_{S^c}. Since $K \subseteq K_S$, there is a surjective homogeneous submersion

$$G/K \longrightarrow G/K_S. \tag{33}$$

A fine cut makes ρ product across $S|S^c$ exactly when its image lies in the corresponding bipartite product-cut locus.

Assume $\rho \neq I/N$, and let β index a nonempty component of the grouped bipartite product-cut locus. Its preimage has the same codimension because Eq. (33) is a fibre bundle. Denote this branchwise codimension by $c_{S,\beta}$. Equation (16) gives

$$c_{S,\beta} \geq 2(d_S - 1)(d_{S^c} - 1), \tag{34}$$

with equality exactly for pure states. There are finitely many bipartitions. For every $\rho \neq I/N$, the cuts making it product across at least one bipartition form a closed subset of zero μ-measure in $\mathcal{T}_{\boldsymbol{q}}$. Almost every cut makes it nonproduct across every bipartition simultaneously.

For a pure state, nonproductness across every bipartition is genuine multipartite entanglement. For a mixed state, it means nonzero total correlation across every bipartition. It does not imply genuine multipartite entanglement. The distinction can be written with quantum mutual information. For density operators τ and ω, write $D(\tau\|\omega) = \mathrm{Tr}[\tau(\log\tau - \log\omega)]$ for quantum relative entropy. At a cut UK,

$$I_U(S : S^c) = D(\rho_U \| (\rho_U)_S \otimes (\rho_U)_{S^c}), \tag{35}$$

which vanishes exactly at bipartite product cuts.

7 Scope

The classified condition is exact product form,

$$\rho_U = \rho_{U,1} \otimes \cdots \otimes \rho_{U,m}. \tag{36}$$

A mixed state is separable when it is a convex combination of product states. It can be separable without satisfying Eq. (36). Every density operator is separable for at least one suitably chosen cut, since an eigenbasis can be mapped to a product basis and the resulting diagonal state is a mixture of product projectors [5]. Some mixed states other than I/N can also remain separable under every global unitary. Absolute separability has a nontrivial spectral geometry of its own [16, 17].

Changing a cut is mathematically equivalent to conjugating the state globally while holding one reference cut fixed. This equivalence does not make the corresponding global unitary a physically available dynamics. The invariant measure on $\mathcal{T}_{\boldsymbol{q}}$ is likewise not a prior over laboratories or apparatus configurations.

Exact independent product decomposition is an exceptional relation between a fixed state and a cut. The theorem does not select a physical cut, identify a spacetime separation, modify Bell local causality, derive a signaling channel, or identify the physical bearer of the global state.

Data availability

No data were generated or analyzed in this work.

Code availability

No code was used to obtain the mathematical results.

Competing interests

The author declares no competing interests.

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